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The algebraic structure of the vacuum Riemann tensor

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Abstract. A $(2j+1)$-spinor formalism is used to discuss the Bel–Petrov–Penrose classification of the Weyl conformal tensor. A convenient pictorial representation of this classification is presented in the form of a series of intersecting manifolds nested in a four-dimensional projective space. The relation to other formalisms is considered briefly.

1. Introduction

An invariant classification of the possible types of space–time is the object of the investigation of the algebraic structure of the Riemann–Christoffel curvature. Apart from being an interesting mathematical problem such a classification is of more 'practical' interest and has found use in the theory of gravitational radiation (e.g. Pirani 1965, 1962) and in the discussion of exact solutions of Einstein's equations (e.g. Ehlers and Kundt 1962).

There are many works on this subject and we shall not attempt to give an exhaustive bibliography. We hope, however, that our references are reasonably comprehensive. The present work will consist essentially of a review of the results obtained by other workers but couched in a somewhat different mathematical language, viz. that of the $(2j+1)$-spinor calculus. This has been discussed earlier (Dowker and Goldstone 1968, to be referred to as GAR, and references therein), and will not be reviewed here. We have, however, to consider the question of $(2j+1)$-spinor algebra in curved space and this is done in the next section. This development is that of our earlier work (Dowker and Dowker 1966, to be referred to as PAS), the notation of which we use now.

In § 3 the question of canonical forms is introduced and the special case of spin 2 is particularly dealt with. This yields the Petrov–Penrose classification (e.g. Pirani 1965). A geometrical picture of this last classification is given in § 4. The concept of principal null spinor is discussed in § 5, and in § 6 the relation of the present formulation to that of earlier authors is considered. Finally in § 7 we present some general comments on the pros and cons of our treatment.

2. $(2j+1)$-spinors in Riemannian space–time, $(2j+1)$-adic components

Since a Riemannian space is pointwise flat we can take the whole development of GAR to be valid at each point of the space independently. To be more precise, consider the group G_P of homogeneous automorphisms of the flat tangent space at P, i.e. the transformations

$$dx^{\mu} \to dx'^{\mu} = \Lambda^{\mu}_{\cdot\nu}\, dx^{\nu} \quad \text{at} \quad P$$

which preserve the tangent scalar product at P,

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = g_{\mu\nu}\, dx'^{\mu}\, dx'^{\nu}.$$

G_P is isomorphic to the homogeneous Lorentz group and we can hence talk about the corresponding $(2j+1)$-spinor representations of G_P which are denoted by $\phi^{(j)}_M$. Under general coordinate transformations these spinors transform as scalars.

If, as we may, we take the $\phi^{(j)}_M$ not to transform under G_P (see PAS, § 2) then we can introduce local Lorentz transformations (which are to be distinguished from G_P) as they are usually defined between Cartesian frames by using the spinor analogue of the tetrad formalism. We choose $2j+1$ base $(2j+1)$-spinors, $\zeta^{m}_{M}(j)$, where the small index just lab ϵ each spinor and runs from $+j$ to $-j$ like the big index. These base spinors satisfy the conditions

$$\begin{aligned} \zeta^m_M \zeta^M_{m'} &= \delta^m_{m'} \\ \zeta^m_M \zeta^{M'}_m &= \delta^{M'}_M. \end{aligned} \tag{1}$$

Small indices are raised and lowered by $\mathring{C}$ the flat space spinor metric and the large indices by C (see GAR and PAS). The ζ also satisfy the requirements

$$\zeta^l_L \zeta^m_M \zeta^n_N \begin{pmatrix} j & j & j \\ l & m & n \end{pmatrix}^0 = \begin{pmatrix} j & j & j \\ L & M & N \end{pmatrix}, \text{ etc.} \tag{2}$$

where $\begin{pmatrix} j & j & j \\ l & m & n \end{pmatrix}^0$ is the usual, i.e. flat space, $3j$ symbol and $\begin{pmatrix} j & j & j \\ L & M & N \end{pmatrix}$ its generalization to curved space (see M. Goldstone, unpublished). In general, objects with only small indices are the flat space ones and the cipher can hence be left off.†

Any $(2j+1)$-spinor ϕ_M can be expanded in the ζ thus

$$\phi_M = \phi_m \zeta^m_M. \tag{3}$$

We may call the coefficients ϕ_m the $(2j+1)$-adic components of ϕ (cf. Newman and Penrose 1962). They are invariant under spin basis changes, general coordinate transformations and the operations belonging to G_P. The covariant derivative of such quantities is thus the same as the ordinary derivative.

The conditions (1) and (2) imply that

$$\begin{aligned} \phi_M \phi^M &= \phi_m \phi^m \\ \phi^L \phi^M \phi^N \begin{pmatrix} j & j & j \\ L & M & N \end{pmatrix} &= \phi^l \phi^m \phi^n \begin{pmatrix} j & j & j \\ l & m & n \end{pmatrix} \end{aligned} \tag{4}$$

etc.

If we now consider a change of the $(2j+1)$-ad basis,

$$\zeta'^m_M = A^m_{\cdot n} \zeta^n_M$$

so that ζ' also satisfies (1) and (2) then by the theorems of Ostrowski (1919) and of Ostrowski and Schur (1922) (see GAR) $A^m_{\cdot n}$ must equal $\overline{\mathscr{D}^{(j,0)}}_{mn}$, a Lorentz group representation matrix. The relation between the old and new $(2j+1)$-adic components of ϕ is determined by

$$\phi_M = \phi_m \zeta^m_M = \phi'_m \zeta'^m_M$$

and is therefore given by

$$\phi'_m = \mathscr{D}^{(j,0)}{}_{mn} \phi_n$$

where

$$\mathscr{D}^{(j,0)}_{mn} = [\exp(\mathrm{i}\epsilon_{\mu\nu} \mathring{J}^{\mu\nu})]^{\cdot n}_{m}$$

is the usual Lorentz transformation matrix, $\mathring{J}^{\mu\nu}$ being the flat space generators (see PAS). Thus, the change of $(2j+1)$-ad basis can be called a (local) Lorentz transformation and it is clear that we can now use, with no changes, the formalism of GAR if all quantities are referred to flat space.

The particular case in which we are interested is that of spin 2 since this corresponds to gravitational theory in the following way.

When $R_{\mu\nu} = 0$ (Einstein's vacuum field equation for gravitation) the Riemann–Christoffel curvature tensor reduces to the Weyl conformal tensor, $C_{\mu\nu\alpha\beta}$. This latter has

† The significance here of the terms flat space and curved space is simply that in flat space the spinor basis is chosen to be the same (say spherical) at every point, whereas in curved space we allow the basis to vary from point to point. The $3j$ symbols, for example, are then functions of position.

all the symmetries of the curvature tensor but in addition is traceless and so belongs to the (2, 0) irreducible representation of the local Lorentz group. We group the ten independent components of $C_{\mu\nu\alpha\beta}$ into a 5-spinor ϕ (using the $A^{\mu\nu}$ spinor tensors) discussed in GAR. Thus we have

$$\phi_M = (A^{\mu\nu\alpha\beta})_M C_{\mu\nu\alpha\beta} \tag{5}$$

and the pentad components of ϕ are of course given by

$$\phi_m = \zeta_m^M \phi_M.$$

It is clear that the ϕ_m are the components of the spinor obtained by an equation like (5) from the *tensor* tetrad components of $C_{\mu\nu\alpha\beta}$ using flat space $A^{\mu\nu\alpha\beta}$. Under rotations of the tensor tetrad these latter do not transform while the tetrad components of $C_{\mu\nu\alpha\beta}$ do. We then see that these rotations correspond to our local Lorentz transformations since under them ϕ_m clearly suffers such transformations.

3. Canonical forms and the classification

The question of canonical forms is now easily expressed. We seek by an appropriate local Lorentz transformation to make vanish as many components as possible of a given ϕ_m. The possible ϕ's then divide into various types or classes according to whether a particular canonical form is possible or not. The inequivalent canonical forms thus provide an invariant classification. We should mention here a small point concerning the nature of a 'canonical' form which is exemplified by the case of spin 1, e.g. electromagnetism. It is well known that a Lorentz transformation applied to the electromagnetic field is just a complex orthogonal transformation (rotation) of the 'vector' or better 3-spinor, $\boldsymbol{H}-\mathrm{i}\boldsymbol{E}$. By a suitable rotation of the axes it is obvious that we can make all but one of the components of this vector zero. This will be a canonical form for this case. In using the particular combination $\boldsymbol{H}-\mathrm{i}\boldsymbol{E}$ we have chosen the spinor metric to be unity. However, in angular momentum analyses it is more usual to use the isotropic basis in which the three components of the 3-spinor are given by

$$\begin{aligned}\phi_1 &= \frac{1}{\sqrt{2}}\{(H_x+E_y)+\mathrm{i}(H_y-E_x)\}\\ \phi_0 &= H_z-\mathrm{i}E_z\\ \phi_{-1} &= \frac{1}{\sqrt{2}}\{(H_x-E_y)-\mathrm{i}(H_y+E_x)\}.\end{aligned}$$

In this basis the canonical forms obtained in the orthogonal basis used previously are given by

$$\begin{array}{ccc} \dfrac{1}{\sqrt{2}}\begin{pmatrix} H-\mathrm{i}E \\ 0 \\ H-\mathrm{i}E \end{pmatrix}, & \dfrac{1}{\sqrt{2}}\begin{pmatrix} E+\mathrm{i}H \\ 0 \\ -E-\mathrm{i}H \end{pmatrix}, & \begin{pmatrix} 0 \\ H-\mathrm{i}E \\ 0 \end{pmatrix} \\ \text{if } (\boldsymbol{H}-\mathrm{i}\boldsymbol{E})\|\boldsymbol{x} & \text{if } (\boldsymbol{H}-\mathrm{i}\boldsymbol{E})\|\boldsymbol{y} & \text{if } (\boldsymbol{H}-\mathrm{i}\boldsymbol{E})\|\boldsymbol{z} \end{array} \tag{6}$$

and it will be seen that in two of the cases *two* components of the spinor are non-zero, although there is a relation between them. The three spinors in (6) do not represent distinct canonical forms. The choice between them is the same choice we have in writing a quadratic form either as the sum of two squares or as the product of two linear factors, and this leads us onto the general treatment of canonical forms.

As described in GAR, associated with a $\phi_m(j)$ is a quantic of order $2j$ in some parameter t, the ϕ_m being the coefficients of this quantic. The search for the canonical forms of ϕ is the same as the search for the canonical forms of the associated quantic. This is a classic mathematical problem and is usually discussed using the techniques of invariant theory (see Turnbull 1960, p. 265) and we can, in fact, simply read off the results for the particular case in which we are interested, assuming, of course, that it has been discussed.

Essentially the classification is based on the coincidences amongst the $2j$ values of t obtained by solving the quantic equation given by setting the associated quantic equal to zero.

Thus, for spin 2, we have the possible root distributions [4], [31], [22], [211], [1111] where, for example, [22] means that we have two pairs of equal roots. The conditions for the existence of a particular root distribution are relations amongst the irreducible concomitants of the quantic. We give here the result for the binary quartic (spin 2) which is just the Petrov–Penrose classification of the vacuum Riemann tensor (Penrose 1960):

[1111]	I	$I^3-27J^2 \neq 0$
[211]	II	$I^3-27J^2 = 0, \quad I \neq 0$
[22]	D	$T_m = 0$
[31]	III	$I = 0 = J$
[4]	N	$H_m = 0$
	0	$\phi_m = 0$

where, specifically, the set of irreducible concomitants is given by

$$I = \phi^m \phi_m$$

$$J = \left(\frac{35}{54}\right)^{1/2} \begin{pmatrix} 2 & 2 & 2 \\ l & m & n \end{pmatrix} \phi^l \phi^m \phi^n$$

$$H_m = \begin{pmatrix} 2 & 2 & 2 \\ m & l & n \end{pmatrix} \phi^l \phi^m, \qquad H = \left(\frac{35}{3}\right)^{1/2} \xi^m(2) H_m$$

$$T_m = \begin{pmatrix} 3 & 2 & 2 \\ m & l & n' \end{pmatrix} \begin{pmatrix} n' & 2 & 2 \\ 2 & k & n \end{pmatrix} \phi^l \phi^k \phi^n, \qquad T = \xi^m(3) T_m.$$

Here, $\xi(j)$ is a null $(2j+1)$-spinor satisfying

$$\begin{pmatrix} 2j-2 & j & j \\ l & m & n \end{pmatrix} \xi^m(j) \xi^n(j) = 0 \tag{7}$$

(see GAR, § 2). It can be checked, using the known values of the $3j$ symbols, that these expressions are identical, apart from multiplicative factors, to the (coefficients of the) concomitants as given in the standard works (e.g. Salmon 1878, Elliott 1895, Burnside and Panton 1904).

We note, in addition, that there are two subtypes of type I fields, viz. those, I_1, for which $J = 0$ and those, I_2, for which $I = 0$.

The corresponding canonical forms can now be found by trial and error or be read off from the standard works. They are

$$\mathrm{I} = \begin{pmatrix} a \\ 0 \\ c \\ 0 \\ a \end{pmatrix}, \qquad 4c^3 - cI + J = 0, \qquad a^2 = I - 3c^2;$$

$$\mathrm{I}_1 = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \\ a \end{pmatrix}; \qquad \mathrm{I}_2 = \begin{pmatrix} a \\ 0 \\ 0 \\ a \\ 0 \end{pmatrix}, \qquad a = (-J)^{1/3}.$$

$$\mathrm{II} = \begin{pmatrix} a \\ 0 \\ c \\ 0 \\ 0 \end{pmatrix}, \qquad c = (\tfrac{1}{3}I)^{1/2}, \qquad ac = H_2; \quad \text{or} \quad \begin{pmatrix} 0 \\ b \\ c \\ 0 \\ 0 \end{pmatrix}, \qquad b = (-H_2)^{1/2}.$$

$$\mathrm{III} = \begin{pmatrix} 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad b = (-H_2)^{1/2}.$$

$$\mathrm{D} = \begin{pmatrix} 0 \\ 0 \\ c \\ 0 \\ 0 \end{pmatrix}, \qquad c = (\tfrac{1}{3}I)^{1/2}.$$

$$\mathrm{N} = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad a \text{ arbitrary}.$$

Some other equivalent forms can be found.

4. A geometrical representation of the classification

One simple and direct geometrical picture of the classification is one that we have already given and is that of the distribution of the roots of the quantic. These we can think of as being arranged on the complex (projective) line. We recognize the fact that what we are giving here is simply one of the many possible geometrical interpretations of binary invariant theory (see, e.g., Turnbull 1960, p. 280, Grace and Young 1903). In GAR, § 2, we discussed a rather attractive and suggestive representation in which the roots of the quantic determine $2j$ points on a norm (unicursal) curve C_{2j}. These points are the feet of the osculating $(2j-1)$-flats from the point in the $2j$-dimensional complex projective space whose homogeneous coordinates are the $2j+1$ components ϕ_m. When the object ϕ belongs to a particular type it means that the corresponding point has some projectively invariant relationship with the norm curve (e.g. Grace 1928).

For spin 1 the classification is simple and is just that into null and non-null fields familiar from electromagnetic theory. Null fields correspond to points on, and non-null fields to points off, the norm curve C_2 which in this case is a conic. The situation for spin 2 is similar but, of course, more complicated. The geometrical details have been worked out by Brusotti (1904) (see Telling 1936 for a detailed account and interesting results) and we shall content ourselves here with a statement of the answers. (A more complete discussion will be found in the Manchester thesis of one of us (J.A.R.).) If $H_m = 0$, ϕ lies on C_4. If $I = 0 = J$, ϕ lies on a tangent to C_4. If $I^3 = 27J^2$, ϕ lies on a tangent plane to C_4, and if $T_m = 0$, ϕ lies on two tangent planes to C_4. If $J = 0$, ϕ lies on a chord of C_4.

To determine the embedding properties of the manifolds determined by these various conditions we note the following results, which follow by symbolic or ordinary methods:

(i) Any two of $I = 0$, $J = 0$, $I^3 = 27J^2$ implies the third.
(ii) $T_m = 0$ implies $I^3 = 27J^2$.
(iii) $I = 0, J = 0$ and $T_m = 0$ implies $H_m = 0$ by the syzygy $T^2+4H^3 = \Phi^2HI-\Phi^3J$.
(iv) $H_m = 0$ implies $I = 0$, $J = 0$ and $T_m = 0$.
These results give the scheme according to which the five spaces

$$\mathrm{N}\ (\text{i.e. } H_m = 0), \qquad \mathrm{III}\ (I = 0 = J), \qquad \mathrm{D}\ (T_m = 0), \qquad \mathrm{II}\ (I^3 = 27J^2,\ I \neq 0)$$

and I (general or I_1 or I_2) are embedded in one another. This scheme is pictured in figure 1.

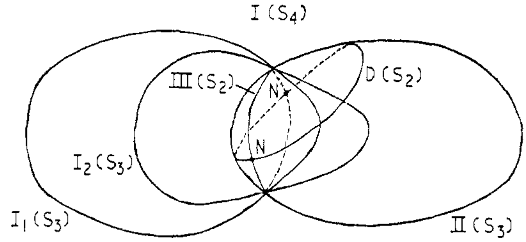

Figure 1. The embedding relationships of the various Petrov–Penrose types. The dimensions of the manifolds have been reduced by one so that, e.g., the S_3 are to be pictured as 'egg-shells' in three-dimensional space (representing S_4) which intersect along the curve III. This curve intersects the curve D, which lies on the surface II, in the points N (representing an S_1). S_n is a complex projective space of dimension n.

In the diagram shown, the dimensions of the manifolds have been diminished by one for representational purposes and the hypersurfaces S_3 are to be thought of as egg-shells embedded in a three-dimensional space (representing S_4) so that III is to be pictured as a closed curve cutting D in N, which is represented by the two points marked N in the figure.

From this structure follows directly the result of Białas (1963) on the continuity properties of the Petrov–Penrose classification. Our argument is most easily appreciated by reference to the spin 1 case.

If, at a space–time point P, the field is non-null then the representative point in S_2 lies *off* the base conic C_2. This means that there exists a finite-size neighbourhood around the point in S_2, and hence in space–time around P, in which the field is *everywhere* non-null also. If the field at P happens to be null then the neighbourhood of the corresponding point in S_2 contains points both on and off the conic. In other words, combining both cases, at a neighbouring space–time point the field can only be of equal or less algebraic speciality.

A simple extension of this topological argument produces the same answer for the spin 2 case with the added complication that type III cannot unspecialize into type D since the manifold III does not contain the manifold D. Further, types II and D cannot unspecialize into types I_1 or I_2 except in special cases.

These results also follow, perhaps more directly, but less attractively in our view, from a consideration of the coincidences of four points on a line.

5. Principal null spinors

Principal null spinors ξ belonging to the field ϕ are defined by the equation, in a $(2j+1)$-ad basis,

$$\tilde{\xi}\tilde{\mathring{C}}\phi = 0 \tag{8}$$

where ξ is a null spinor, i.e. satisfies (7). By the standard theorems of algebra there will be, in general, $2j$ spinors ξ satisfying (8). With each of these we can associate, as explained in GAR, a null space–time vector. These vectors are the principal null vectors as usually defined (Penrose 1960, 1965).

If the parameter from which the null spinor ξ in (8) is constructed is denoted by t then equation (8) is just the quantic equation referred to in § 3. The $2j$ principal null spinors are thus obtained by substituting the $2j$ roots of this equation into ξ. Instead of talking about coincidences amongst the roots we can then talk about coincidences amongst the principal null spinors (or vectors).

No more will be said about principal null spinors here except that, just like principal null 2-spinors and vectors, they can be used to give an expression for the field ϕ. Thus for type I fields (the most general ones) we have

$$\phi_m \propto \epsilon_{mm_1 \ldots m_{2j}} \xi^{m_1} \ldots \xi^{m_{2j}}$$

where $\epsilon_{m \ldots r}$ is a Levi-Civita permutation symbol (cf. Veblen and von Neumann 1936).

6. Relation to other formalisms

In the purely tensor approach as used by Sachs (1961), for example, the independent (invariant) components of the Weyl tensor are 'projected out' using the null tetrad formalism (see, for example, Pirani 1965).

Four vectors k, m, t, $\bar{t}$ are constructed with the properties

$$k^\mu m_\mu = \bar{t}^\mu t_\mu = 1, \qquad k^\mu k_\mu = m^\mu m_\mu = t^\mu t_\mu = k^\mu t_\mu = m^\mu t_\mu = 0.$$

Then we make the combinations (Sachs 1961)

$$M^{\mu\nu} = 2k^{[\mu} m^{\nu]} + 2\bar{t}^{[\mu} t^{\nu]}, \qquad V^{\mu\nu} = 2k^{[\mu} \bar{t}^{\nu]}, \qquad U^{\mu\nu} = 2m^{[\mu} t^{\nu]}.$$

These form, essentially, a 3-spinor, or bivector, basis. Now the further five combinations are made,

$$V_{\mu\nu} V_{\alpha\beta}, \qquad V_{\mu\nu} M_{\alpha\beta} + V_{\alpha\beta} M_{\mu\nu}, \qquad M_{\mu\nu} M_{\alpha\beta} + U_{\mu\nu} U_{\alpha\beta} + V_{\mu\nu} U_{\alpha\beta}$$
$$U_{\mu\nu} M_{\alpha\beta} + M_{\mu\nu} U_{\alpha\beta}, \qquad U_{\mu\nu} U_{\alpha\beta}$$

which form a 5-spinor basis. If we use the quantities $A_M^{\mu\nu\alpha\beta}(2)$ discussed in GAR we have the explicit relation

$$\zeta_M^2 = A_M^{\mu\nu\alpha\beta}(2) V_{\mu\nu} V_{\alpha\beta}, \text{ etc.}$$

So that Sachs' definition of ϕ^2, etc.,

$$\phi^2 = C^{\mu\nu\alpha\beta} V_{\mu\nu} V_{\alpha\beta} = \phi^M A_M^{\mu\nu\alpha\beta}(2) V_{\mu\nu} V_{\alpha\beta}$$
$$= \phi^M \zeta_M^2, \text{ etc.}$$

agrees with ours, up to a factor.

In the approach of Debever (1964) and Geheniau (1957) (see also Synge 1964, Peres 1962, Buchdahl 1966, Cahen *et al.* 1967) use is made of bivectors (or 3-spinors) which are, so to speak, the fundamental 'building units' for quantities of integral spin. The classification devolves upon the distribution of the eigenvalues of a 3×3 complex matrix W obtained from the Weyl tensor. Using the notation of GAR we have for this matrix

$$W_{rs} \propto A_r^{\mu\nu}(1) A_s^{\alpha\beta}(1) C_{\mu\nu\alpha\beta}.$$

It is shown in Debever (1964, § 17) that the problem of the various Petrov types is, geometrically, just that of the relative positions (i.e. the distribution of the intersections) of two conics. In our language we would say that these two conics are situated in 3-spinor (i.e. (1, 0)) space. They are given by the equations

$$\phi^r(1)\phi_r(1) = 0 = \phi^r(1) W_{rs} \phi^s(1).$$

This result follows from our discussion in § 3 by Burnside's somewhat elementary theorem that solving the quartic equation is equivalent to solving two simultaneous quadratic equations.

Turning briefly to the 2-spinor technique (Penrose 1960, Witten 1959), instead of $\phi(2)$ or $C_{\mu\nu\alpha\beta}$ the symmetric fourth-rank 2-spinor ψ_{abcd} is employed and the quartic $\xi^m \phi_m$ is written as $\psi_{abcd}\xi^a\xi^b\xi^c\xi^d$, ξ^m being a (null) 5-spinor formed from four ξ^a 2-spinors. It will be appreciated that our discussion of the classification is exactly the same as that given by Penrose (1960); only our notation and geometrical representation are different.

7. Discussion and conclusion

One advantage of the $(2j+1)$-spinor formalism is that it allows us to make use of the standard results of algebraic theory rather more immediately than the other formalisms and one concentrates on the actual algebraically independent degrees of freedom of the fields. Again, as we have mentioned previously, the various formulae are compact in appearance. For example the expression for the cubinvariant J in terms of a $3j$ symbol is, we think, quite neat. Also we think that the pictorial representation of the Petrov–Penrose classification given in § 4 is rather attractive. Unfortunately one pays a price for this elegance in that in any particular calculation, if one uses the $(2j+1)$-spinors, specific algebraic properties of the $3j$ symbols may be needed, and it is not clear where these will come from unless we go back to the basic 2-spinor algebra. We do not, therefore, claim that a 'universal' use of our formalism always leads to significant computational advantages.

An advantage of the 2-spinor formalism, and therefore a disadvantage of the $(2j+1)$-formalism, is the simple relation between 4-vectors and 2-spinors. This shows up, for example, in the expression $\psi_{abcd} = \alpha_{(a}\beta_b\gamma_c\delta_{d)}$ for the curvature spinor in terms of the principal null 2-spinors α, β, γ, δ. The principal null vectors of the curvature are then just the null vectors associated with α, β, γ, δ. Again, because space–time is most conveniently (presumably) described in terms of a 4-vector the usual dynamical equations for the fields involve a derivative ∇_μ which, in some ways, seems to stick out and 'spoil' the $(2j+1)$-formalism.

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